Spectral fluctuation characterization of random matrix ensembles through wavelets

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Abstract.

   A recently developed wavelet based approach is employed to characterize the scaling behavior of spectral fluctuations of random matrix ensembles, as well as complex atomic systems. Our study clearly reveals anti-persistent behavior and supports the Fourier power spectral analysis. It also finds evidence for multi-fractal nature in the atomic spectra. The multi-resolution and localization nature of the discrete wavelets ideally characterizes the fluctuations in these time series, some of which are not stationary.

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Recently, a fundamentally different approach to the study of random matrix ensembles has been taken [1, 2, 3, 4]. The cumulative of the unfolded spectra for a variety of ensembles has been treated as a time series, on which a Fourier spectral analysis has been carried out. Very interestingly, this reveals a ubiquitous $1/f$ power law behavior in the Fourier domain, which has been taken as the defining characteristic of these random matrix ensembles. In this context, a number of tools, earlier employed for the analysis of various time series showing self-similar behavior, can be used for the characterization of the above data. The well-known methods like rescaled range analysis [5], structure function method [6], wavelet transform modulus maxima [7] etc., although powerful, have been found to be wanting, when applied to non-stationary time series. Recently, we have developed a discrete Daubechies wavelets [8] based approach [9, 10], which is well suited to remove local trends in any time series and faithfully characterize the fluctuations. It is complimentary to the well studied multi-fractal detrended fluctuation analysis (MFDFA) [11, 12, 13, 14, 15, 16], which uses appropriate polynomial fits in local windows for extracting fluctuations. It is worth mentioning that, the multi-resolution ability of the wavelets naturally makes them ideal for the analysis of time series showing self-similar behavior [17].

   This letter studies the scaling and correlation behavior and multi-scaling properties of energy fluctuations in random matrix ensembles and atomic systems. The primary

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motivation for this comes from the aforementioned Fourier spectral analysis, which is known to yield the Hurst exponent H, with $\alpha = 2H + 1$, α being the exponent of the power law decay. Since, H is related to the second moment, it is quite natural to ask the nature of the higher moments, which determine the multi-fractal characteristic of the time series. For this purpose, we make use of a discrete wavelet based approach, developed by us earlier [9, 10]. In this approach, after removal of the local trend over a given window, the fluctuations are extracted. The wavelets naturally provide a number of windows of different sizes, to extract fluctuations at various scales. Using these, the fluctuation function [18] is then calculated, which yields the mono or multi-fractal nature of the time series, when studied in a log-log plot. We further test the self similar behavior of the above time series through MFDFA for completeness. Our results on Hurst exponent matches with Fourier spectral analysis.

It is found that certain atomic systems exhibit multi-fractal behaviour. It is observed that removing the eigenvalues of the localized states, which are quantum analogue of classical unstable orbits, from the energy level data does not alter the nature of the self-similar behavior, although it affects the Hurst exponent value. The identification and quantification of localized states are discussed in Ref.[19]. Study of multi-fractal behaviour has been carried out earlier in the context of various atoms and ions. Methods like, box-counting [20] and correlation sum have been employed [21, 22]. Keeping in mind the possible non-stationary nature of the data, as well as efficacy of the wavelet based approach for extracting fluctuations, we have used the same here for studying the self-similar nature of the fluctuations.

We give a brief description of the wavelets before proceeding to the analysis of time series related to various random matrix ensembles [23] and atomic level data [24]. Results and discussions pertaining to the above analysis are then presented. One observes mono-fractal behavior for the time series associated with random matrix ensembles. Two atomic energy level time series show multi-fractal behavior and another reveals mono-fractal character. We summarize and conclude in the final section, after pointing out several directions for future work.

The discrete wavelets provide complete and orthonormal basis functions, starting from the father wavelet $\phi(t)$ (scaling function) and mother wavelet $\psi(t)$ [8,21][8, 25]. These functions necessarily satisfy, $\int \phi(t)dt = A$ and $\int \psi(t)dt = 0$, where A is constant. Scaling and translation of wavelets lead to $\psi_{j,k} = 2^{j/2}\psi(2^j t - k)$, which obey the orthogonality conditions: $\langle \phi_{j,k}|\psi_{j,k}\rangle = 0$ and $\langle \psi_{j,k}|\psi_{j',k}\rangle = \delta_{j,j'}\delta_{k,k'}$. Any signal belonging to L^2 can be written in the form,

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \ \phi_k(t) + \sum_{k=-\infty}^{\infty}\sum_{j=0}^{\infty} d_{j,k} \ \psi_{j,k}(t) \ , \tag{1}$$

where $c_k's$ are the low-pass coefficients and $d_{j,k}'s$ are the high-pass coefficients. The Daubechies family of wavelets are made to satisfy vanishing moment conditions: $\int dt \, t^m \psi_{j,k}(t) = 0$. This makes them ideal to isolate polynomial trends from fluctuations. We make use of the discrete wavelets from Daubechies family for our analysis of energy

level fluctuations.

Consider E_i, $i = 1, 2, 3..., n + 1$, the discrete energy levels of random matrix ensembles or atomic systems, represented as a discrete time series in the form E_i as shown in Figs. 1 and 2. The trend and fluctuations from the integrated level density are separated by the higher order polynomial fitting: $N(E) = N_t(E) + N_f(E)$. The unfolded energy spectra $\bar{E}_i$ is obtained through the transformation $N_t(E_i) = \bar{E}_i$. The fluctuations of the energy level spacings are, $e_i = \bar{E}_{i+1} - \bar{E}_i$, $i = 1, 2, 3, ..., n$, where $< e_i >= 1$. The time series of energy level fluctuations is given by,

$$\delta_m = \sum_{i=1}^{m}(e_i - < e_i >); \quad m = 1, ..., n. \tag{2}$$

The power spectrum of the times series δ_m is calculated for various random matrix ensembles and atomic energy levels of Nd, Pm and Sm. It should be noted that, these have 60, 61 and 62 electrons respectively. The corresponding number of active valence electrons are 6, 7 and 8; the complexity increases as the number of active valence electrons increases. The random matrix ensembles studied are Gaussian orthogonal ensemble (GOE), Gaussian unitary ensemble (GUE), Gaussian diagonal ensemble (GDE) and Gaussian symplectic ensemble (GSE). The cumulative of the unfolded time series are shown in Fig.1 and Fig.2, for the atomic systems and random matrix ensembles respectively. Regarding the details of the atomic system, interested readers are referred to Ref.[24]

Fourier spectral analysis of all the data sets yields, $S(k) \approx \frac{1}{k^\alpha}$, with $\alpha \sim 1$. This is in agreement with the results of Relaño et. al.,, which found that the energy spectra of quantum systems exhibiting classical chaos are characterized by $1/f$ behavior. We carried out the spectral analysis using fast Fourier transform [26], and ensemble averaging was employed in finding the exact slope α.

We now proceed to the study of the scaling properties of these time series through wavelet based fluctuation analysis. We make use of wavelets from Daubechies family for characterization, since these naturally remove polynomial trends from data sets. The fractal nature of the time series is revealed through the study of the fluctuation function. In discrete wavelet transform it is well-known that, a given signal belonging to L^2 space can be represented in a nested vector space spanned by the scaling functions alone. This basic requirement of multi-resolution analysis can be formally written as [?],

$$... \subset \nu_{-2} \subset \nu_{-1} \subset \nu_{-0} \subset \nu_1 \subset \nu_2... \subset L^2, \tag{3}$$

with $\nu_{-\infty} = 0$ and $\nu_\infty = L^2$. This provides a successive approximation of a given signal in terms of low-pass or approximation coefficients. It is clear that, the space that contains high resolution signals will also contain signals of lower resolution. The signal or time series can be approximated at a level of ones choice, for use in finding the local trend over a desired window. The fluctuations can then be obtained by subtracting the above trend from the signal. We have followed this approach for extracting the fluctuations, by elimination of local polynomial trends through the Daubechies wavelets.

We compute the fluctuation function in order to ascertain the self-similar nature of the time series. The qth order fluctuation function, $F_q(s)$ is obtained by squaring and averaging fluctuations over all segments:

$$F_q(s) \equiv \{\frac{1}{2M_s} \sum_{b=1}^{2M_s} [F^2(b,s)]^{q/2}\}^{1/q}. \tag{4}$$

Here 'q' is the order of moments that takes any real values. The above procedure is repeated for variable window sizes for different value of q (except $q = 0$). The scaling behavior is obtained by analyzing the fluctuation function,

$$F_q(s) \sim s^{h(q)}, \tag{5}$$

in a logarithmic scale for each value of q. If the order $q = 0$ logarithmic averaging has to be employed to find the fluctuation function:

$$F_0(s) \equiv exp\{\frac{1}{4M_s} \sum_{b=1}^{2M_s} ln[F^2(b,s)]\}. \tag{6}$$

As is well-known, if the time series is mono-fractal, the $h(q)$ values are independent of q. For multifractal time series, $h(q)$ values depend on q. The correlation behavior is characterized from the Hurst exponent ($H = h(q = 2)$), which varies from $0 < H < 1$. For long range correlation, $H > 0.5$, $H = 0.5$ for uncorrelated and $H < 0.5$ for long range anti-correlated time series. We refer the interested readers to [9, 10] for the details of this approach. The power law manifests itself as a straight line in the log-log plot of $F_q(s)$ versus s for each value of q:

$$F_q(s) \sim s^{h(q)}. \tag{7}$$

For mono-fractal time series, $h(q)$ is constant for all q, whereas for multi-fractal time series h(q) shows non-linear dependence for all q. Here q varies from -10 to $+10$. We have used Db-8 wavelet for capturing fluctuations. The well known Hurst exponent H equals to $h(q = 2)$, which is related to the power spectral analysis by the relation $\alpha = 2H + 1$. Since, the values of the fluctuations are very small, in order to study the same, we integrated the time series by subtracting mean. Through this double integrated time series, the obtained Hurst exponent is $H_1 = H + 1$.

The Hurst exponent calculated from Fourier analysis compares well with the wavelet based fluctuation analysis for correlation behavior. We have also corroborated our results through MFDFA.

The self-similar behavior of a variety of random matrix ensembles and atomic level data have been explored through both discrete wavelets and MFDFA. The scaling behavior corroborates the findings of Fourier analysis. We found mono-fractal behavior for random matrix ensemble time series for which computed Hurst exponents agreed with the results of Relano *et. al.* Very interestingly, for one atomic level data, we observed multifractal behavior for Pm and Nd systems. Sm showed scaling behaviour. Removal of the eigenvalues of the localized states made the Sm system monofractal with strong persistence. These results are shown in Table. 1. In the last column Sm_d
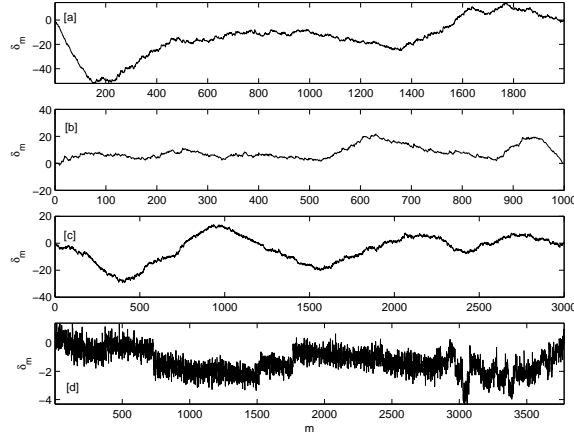
Figure 1. Time series of extracted fluctuations from the unfolded energy spectra of [a] Pm, [b] Nd and [c]and [d] Sm atomic energy levels.
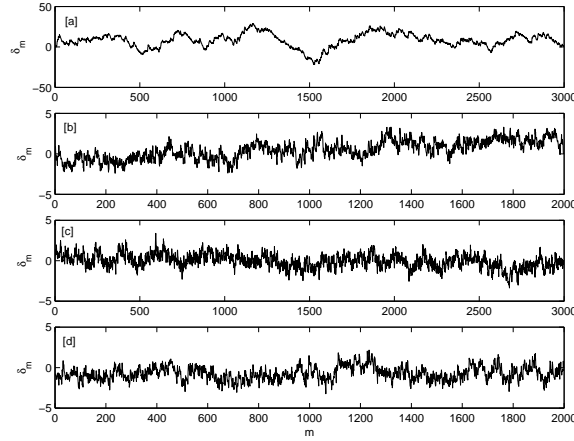
Figure 2. Time series of extracted fluctuations from the unfolded energy spectra of random matrix ensembles [a] GDE, [b] GSE, [c] GOE and [d] GUE.

represents the atomic level data for Sm when the eigenvalues of the localized states have been removed. It is worth noting that, among the three atoms, Sm has the strongest configuration mixing as it has the largest number of active valence electrons. This indicates that the lack of sufficient mixing and effect of localized levels may influence the multifractal nature of the atomic systems.

We intend to study these aspects carefully in future. The nature of correlations in other ensembles which appear in various physical problems also needs investigations. These include embedded Random Matrix ensembles relevant for finite interacting particle systems [27].

References

q	GDE	GSE	GOE	GUE	Pm	Nd	Sm	Sm_d
-10	1.5493	1.1442	0.9930	1.0728	1.4092	1.5257	1.1833	0.9440
-9	1.5445	1.1369	0.9901	1.0678	1.4047	1.5231	1.1854	0.9394
-8	1.5395	1.1292	0.9874	1.0625	1.4004	1.5203	1.1880	0.9347
-7	1.5341	1.1213	0.9848	1.0569	1.3963	1.5172	1.1914	0.9301
-6	1.5286	1.1133	0.9826	1.0512	1.3928	1.5139	1.1955	0.9259
-5	1.5234	1.1052	0.9809	1.0457	1.3904	1.5105	1.2003	0.9224
-4	1.5189	1.0971	0.9797	1.0403	1.3893	1.5071	1.2055	0.9202
-3	1.5155	1.0890	0.9793	1.0353	1.3903	1.5042	1.2107	0.9197
-2	1.5130	1.0808	0.9798	1.0307	1.3938	1.5021	1.2154	0.9213
-1	1.5103	1.0726	0.9811	1.0264	1.3999	1.5010	1.2192	0.9252
0	1.5056	1.0644	0.9830	1.0222	1.4080	1.5006	1.2215	0.9307
1	1.4980	1.0563	0.9852	1.0183	1.4159	1.4987	1.2219	0.9365
2	1.4878	1.0481	0.9872	1.0145	1.4165	1.4916	1.2202	0.9409
3	1.4759	1.0401	0.9888	1.0108	1.3983	1.4759	1.2163	0.9436
4	1.4632	1.0321	0.9894	1.0073	1.3621	1.4533	1.2104	0.9449
5	1.4507	1.0244	0.9892	1.0039	1.3251	1.4289	1.2028	0.9450
6	1.4389	1.0170	0.9880	1.0008	1.2958	1.4070	1.1941	0.9444
7	1.4280	1.0099	0.9861	0.9979	1.2738	1.3892	1.1851	0.9432
8	1.4183	1.0032	0.9836	0.9951	1.2571	1.3751	1.1763	0.9417
9	1.4097	0.9970	0.9806	0.9925	1.2440	1.3640	1.1680	0.9401
10	1.4021	0.9913	0.9775	0.9900	1.2334	1.3551	1.1604	0.9383

Table 1. The $h(q)$ values for different values of q obtained from wavelet based fluctuation analysis. Here $h(q = 2) = H$, is the Hurst scaling exponent.

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